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FINITE BIAXIAL EXTENSION OF PARTIALLY SET PLAIN WOVEN FABRICS†

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Abstract—This paper is concerned with the finite deformation of partially set plain woven fabrics subjected to biaxial stresses applied in the directions of the yarns. The configurations of the yarns in the stress-free state are assumed to consist of combinations of elasticas resulting from finitely deformed cantilevers under end loads. In our analysis, the yarns are treated as elastic curved rods subjected to combined extension and bending. As a result of fiber slippage, the moment-curvature relation of the yarn is regarded as bilinear with ideal Baushinger's effect. Residual stresses are found to exist in the fiber initially. A nonlinear boundary value problem can be formulated for the deformation of the fabric where the interyarn contact force is determined from the compatibility condition of displacements at the crimp for yarns in the warp and the weft directions. The effect of the contact deformation of yarns is included in this analysis. Numerical solutions are found for the problem of a uniaxial extension of the fabric and the problem of a biaxial extension of the fabric with equal stresses.

INTRODUCTION

The plain woven fabric normally consists of interlaced bent yarns in two mutually perpendicular directions. Prior to weaving, the configuration of the yarn is straight. With weaving, residual stresses are introduced to the yarn. The fabric with unrelaxed residual stresses is referred to as a grey fabric. Residual stresses in a fabric can be eliminated through a stress relaxation treatment by means of a moistifying and drying process. When the residual stresses are entirely relaxed, the fabric is called a completely set fabric. In reality, there is always a certain amount of residual stress remaining in the fabric. The fabric with partially relaxed stresses is called a partially set fabric. The tensile property of the fabric can be affected by the presence of the residual stresses.

The initial extensional moduli of grey and completely relaxed fabrics have been investigated by Grosberg and Kedia based on an energy method associated with small deformation[1]. In their analysis, the yarns are considered as inextensible. They concluded that the fabric with residual stresses would have a higher initial extensional modulus. The problem of the finite biaxial extension of completely set woven fabrics has been studied by Huang[2]. In his formulation, the yarns are treated as extensible curved rods with bilinear moment-curvature relation resulting from fiber slippage during flexural deformation of the yarn.

In this paper, we shall generalize the method presented in [2] to the case of partially set plain weaves. Since there is an initial deformation caused by residual stresses, an unloading process can occur in the yarn under the action of external forces. We shall consider that the yarn material has an ideal Baushinger's effect in bending during the unloading process. A nonlinear boundary value problem can be formulated for the initial and the final deformed states of the fabric. Solutions will be obtained by an iterative procedure.

GEOMETRY OF THE YARN

Let us consider a piece of partially set plain woven fabric consisting of interlaced yarns in two mutually perpendicular directions referred to as the warp and the weft directions. The fabric is subjected to biaxial forces in the warp and the weft directions. The problem of a uniaxial extension will be regarded as a special case when the biaxial forces in one direction vanish. In general, the deformation of the fabric can be described by the following three states: (i) the stress-free state, (ii) the initial state prior to the application of external forces and (iii) the final deformed state due to the action of external forces. Note that ther are residual stresses in

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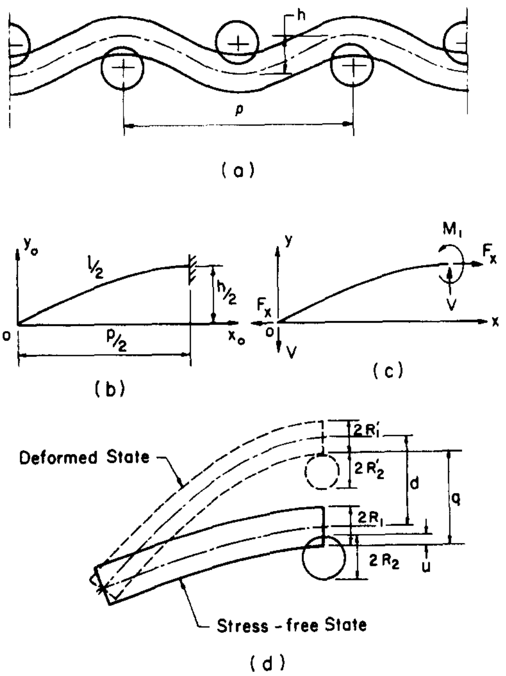

Fig. 1. Geometry of the problem.

the yarn in the initial state. In order to obtain yarns in the stress-free state, it is necessary to separate the yarns from the fabric. The yarns in the warp and the weft directions in the stress-free state would overlap with each other as shown in Fig. 1(a). The center line of the yarn is in the shape of a wave with the wave length $2p$ and the amplitude h. We shall call $2p$ and h as the thread spacing and the crimp height respectively. The arc length of the center line for half of the wave is called the yarn length and is denoted by l. There are points of inflection on the center line of the yarn. The configuration of the yarn is always antisymmetrical with respect to the point of inflection. Note that all points of inflection of yarns in the warp direction lie in a plane P_1 and all points of inflection of yarns in the weft direction lie in another plane P_2. These planes are parallel to each other. The height of the P_1-plane as measured from the P_2-plane is denoted by H. Let us consider a segment of the center line of the yarn in the stress-free state as shown in Fig. 1(b). If we set the origin at the point of inflection, the coordinates of the crimp point would be $(p/2, h/2)$. The depth of the yarn is denoted by $2R$. The center line of the yarn is assumed at the half depth. In the following, we shall consider the yarn in the plane of Fig. 1(a) as the one in the warp direction and the yarn perpendicular to the plane of Fig. 1(a) as the one in the weft direction. We shall use subscripts 1 and 2 to indicate quantities referred to the yarns in the warp and the weft directions. The vertical distance at the crimp between the bottom of the yarn in the warp direction and the top of the yarn in the weft direction is

$$U' = R_1 + R_2 - H - \frac{1}{2}(h_1 + h_2). \tag{1}$$

We may consider U as a measure of the degree of stress relaxation. Larger value of U corresponds to the case of a lesser degree of stress relaxation. For a completely set fabric, the stress-free state and the initial state are identical. Hence $U = 0$ and

$$H = R_1 + R_2 - \frac{1}{2}(h_1 + h_2). \tag{2}$$

In the following, we shall follow Peirce[3] and assume that the geometry of the center line of the yarn between the point of inflection and the crimp in the stress-free state as shown in

Fig. 1(b) is identical to that of an elastica corresponding to a finitely deformed cantilever under an end force. Let us denote the angle of inclination of the tangent at the origin by α and put

$$k = \sin\left(\frac{\alpha}{2} + \frac{\pi}{4}\right) \tag{3}$$

and

$$\rho = \sin^{-1}(2^{-1/2}/k). \tag{4}$$

The crimp height and the thread spacing are respectively

$$h = l\left[1 - 2\frac{E(k) - E(\rho, k)}{K(k) - F(\rho, k)}\right] \tag{5}$$

and

$$p = 2kl\cos\rho/[K(k) - F(\rho, k)], \tag{6}$$

where $K(k)$ and $F(\rho, k)$ are the complete and the incomplete elliptic integrals of the first kind; $E(k)$ and $E(\rho, k)$ are the complete and the incomplete elliptic integrals of the second kind. Let θ_0 be the angles of inclination of the tangent at any point on the center line. We have

$$\cos\theta_0 = 2k\sin\phi(1 - k^2\sin^2\phi)^{1/2}, \tag{7}$$

where ϕ is a parameter related to the abscissa x_0 by

$$\phi = \cos^{-1}\left\{\frac{x_0}{kl}[K(k) - F(\rho, k)]\right\}. \tag{8}$$

Let s_0 be the arc length measured along the center line. The curvature of the center line in the the stress-free state is

$$\frac{d\theta_0}{ds_0} = -\frac{2^{3/2}}{l}[K(k) - F(\rho, k)](\sin\alpha - 2k^2\sin^2\phi + 1)^{1/2}. \tag{9}$$

STRESS ANALYSIS

The yarn is deformed by the residual stresses in the initial state. Further deformation is introduced due to the application of forces F_x and F_z to the yarns in the warp and the weft directions respectively. Since the deformation is antisymmetrical with respect to the point of inflection, the P_1 and P_2-planes must remain stationary during deformation. Again, we shall set the origin at the point of inflection in the deformed yarn. A contact force V and a bending moment M_1 are introduced at the crimp as shown in Fig. 1(c). Let the angle of inclination of the tangent at any point be θ. The axial tension and the bending moment at any point are

$$T = F_x\cos\theta + V\sin\theta \tag{10}$$

and

$$M = F_x y - Vx \tag{11}$$

respectively.

Let AE be the extensional stiffness of the yarn. The axial strain is

$$\epsilon = \frac{1}{AE}(F_x\cos\theta + V\sin\theta). \tag{12}$$

Denote the arc length of the center line of the deformed yarn by s. We have

$$\frac{ds}{ds_0} = 1 + \epsilon. \tag{13}$$

The change in curvature is

$$\kappa = \frac{d\theta}{ds} - \frac{d\theta_0}{ds_0}. \tag{14}$$

By eqns (13) and (14), we have

$$\frac{d\theta}{dx_0} = \frac{1+\epsilon}{\cos\theta_0}\left(\frac{d\theta_0}{ds_0} + \kappa\right), \tag{15}$$

where $d\theta_0/ds_0$ is given by eqn (9) and

$$\frac{1+\epsilon}{\cos\theta_0} = \left[1 + \frac{1}{AE}(F_x \cos\theta + V \sin\theta)\right] / [2k \sin\phi(1 - k^2 \sin^2\phi)^{1/2}]. \tag{16}$$

As a result of the deformation in the initial state, unloading in bending moment can be introduced in the yarn during the application of external forces. We shall consider that the $M-\kappa$ relation is bilinear. Let M_0 be the magnitude of bending moment when fiber slippage begins to take place. The bending stiffness varies from EI to E^*I ($E^* < E$) due to fiber slippage. We shall also assume that the material of the yarn possesses an ideal Baushinger's effect during the unloading process as shown in Fig. 2. It is easy to realize intuitively that if there is any reversal in curvature, it will initiate at the instance of the application of external forces. Thus if the external forces are applied incrementally, the reversal in curvature can occur only at the first increment of external force. Let M_a and κ_a be respectively the bending moment and the change in curvature caused by the residual stresses at the initial state. The $M-\kappa$ relation can be expressed mathematically by the following equations:

$$\kappa = \begin{cases} \dfrac{M}{EI} & \text{for } |M| \geq |M_a| \text{ and } |M| \leq M_0; \\ \dfrac{M_0}{EI} + \dfrac{M - M_0}{E^*I} & \text{for } |M| > |M_a|, |M| > M_0 \text{ and } M > 0; \\ -\dfrac{M_0}{EI} + \dfrac{M + M_0}{E^*I} & \text{for } |M| > |M_a|, |M| > M_0 \text{ and } M < 0; \\ \kappa_a + \dfrac{M - M_a}{EI} & \text{for } |M| \leq |M_a| \text{ and } |M_a - M| \leq 2M_0; \\ \kappa_a - \dfrac{2M_0}{EI} + \dfrac{M - M_a + 2M_0}{E^*I} & \text{for } |M| < |M_a|, |M_a - M| > 2M_0 \text{ and } M_a > 0; \\ \kappa_a + \dfrac{2M_0}{EI} + \dfrac{M - M_a - 2M_0}{E^*I} & \text{for } |M| < |M_a|, |M_a - M| > 2M_0 \text{ and } M_a < 0. \end{cases} \tag{17}$$

By eqns (15)–(17) and (11), we can express $d\theta/dx$ as a function of θ, x and y. Since

$$\frac{dx}{ds} = \cos\theta, \qquad \frac{dy}{ds} = \sin\theta, \tag{18}$$

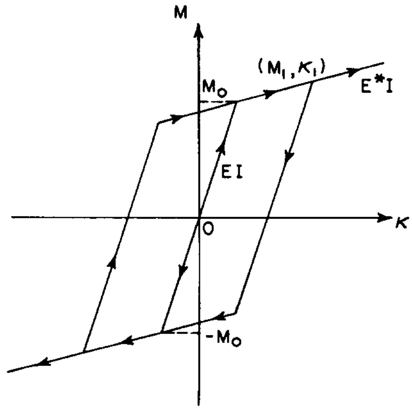

Fig. 2. $M-\kappa$ relation for a material with ideal Baushinger's effect.

we have

$$\frac{dx}{dx_0} = \frac{1+\epsilon}{\cos\theta_0}\cos\theta \tag{19}$$

and

$$\frac{dy}{dx_0} = \frac{1+\epsilon}{\cos\theta_0}\sin\theta. \tag{20}$$

Thus, if we treat x_0 as an independent variable and θ, x and y as dependent variables, eqns (15), (19) and (20) present a nonlinear boundary value problem. The boundary conditions are

$$x(0) = y(0) = \theta(p/2) = 0. \tag{21}$$

In order to solve this problem, we must find the interyarn pressure V. The upward displacement of the center line at the crimp is

$$\begin{aligned} d &= y(p/2) - h/2 \\ &= y(p/2) - \frac{l}{2}\left[1 - 2\frac{E(k) - E(\rho, k)}{K(k) - F(\rho, k)}\right]. \end{aligned} \tag{22}$$

The change in yarn depth $2R' - 2R$ is caused by (i) Poisson's effect and (ii) the contact deformation. The average normal strain across the depth of the yarn due to the contact deformation is assumed to be proportional to the contact force V. Hence

$$\frac{R' - R}{R} = -[\sigma\epsilon(p/2) + \lambda V], \tag{23}$$

where σ is Poisson's ratio and λ is a constant. The upward vertical displacement at the point of contact of the yarns as shown in Fig. 1(d) is

$$\begin{aligned} q &= d - (R' - R) \\ &= d + R[\sigma\epsilon(p/2) + \lambda V]. \end{aligned} \tag{24}$$

The compatibility condition for the displacement at the point of contact for yarns in the warp and the weft directions requires that

$$q_1 - q_2 = U, \tag{25}$$

which leads to

$$d_1 + R_1[\sigma_1\epsilon_1(p_1/2) + \lambda_1 V] + d_2 + R_2[\sigma_2\epsilon_2(p_2/2) + \lambda_2 V] = U. \tag{26}$$

Equation (26) provides an additional condition for the determination of V.

Let us introduce the following dimensionless quantities:

$$\begin{aligned} &\bar{x}_0 = 2x_0/l,\ \bar{y}_0 = 2y_0/l,\ \bar{s}_0 = 2s_0/l,\ \bar{x} = 2x/l,\ \bar{y} = 2y/l,\ \bar{d} = 2d/l, \\ &\bar{R} = 2R/l,\ \bar{\kappa} = l\kappa/2, \bar{\kappa}_a = l\kappa_a/2,\ n = l/(2r),\ v = V/(AE),\ m_0 = M_0 l/(2EI), \\ &m = Ml/(2EI),\ m_a = M_a l/(2EI),\ \nu = E/E^*,\ \bar{\lambda} = \lambda AE, \end{aligned} \tag{27}$$

where r is the radius of gyration of the cross section with respect to the neutral axis. A subscript 1 or 2 can be added to denote the quantities for the yarn in the warp or the weft direction. Let us also introduce the following dimensionless quantities:

$$\beta = l_2/l_1,\ \mu = A_2E_2/(A_1E_1),\ u = 2U/l_1,\ f_x = F_x/(A_1E_1),\ f_z = F_z/(A_1E_1). \tag{28}$$

Our governing equations can be expressed in terms of the dimensionless quantities as

$$\phi = \cos^{-1}\left\{\frac{\bar{x}_0}{2k}[K(k) - F(\rho, k)]\right\}, \tag{29}$$

$$\frac{d\theta_0}{d\bar{s}_0} = -2^{1/2}[K(k) - F(\rho, k)](\sin\alpha - 2k^2\sin^2\phi + 1)^{1/2}, \tag{30}$$

$$m = n^2(f_x\bar{y} - v\bar{x}), \tag{31}$$

$$\bar{\kappa} = \begin{cases} m & \text{for } |m| \geq |m_a| \text{ and } |m| \leq m_0, \\ m_0 + \nu(m - m_0) & \text{for } |m| > |m_a|, |m| > m_0 \text{ and } m > 0, \\ -m_0 + \nu(m + m_0) & \text{for } |m| > |m_a|, |m| > m_0 \text{ and } m < 0, \\ \bar{\kappa}_a + m - m_a & \text{for } |m| \leq |m_a| \text{ and } |m_a - m| \leq 2m_0, \\ \bar{\kappa}_a - 2m_0 + \nu(m - m_a + 2m_0) & \text{for } |m| < |m_a|, |m_a - m| > 2m_0 \text{ and } m_a > 0, \\ \bar{\kappa}_a + 2m_0 + \nu(m - m_a - 2m_0) & \text{for } |m| < |m_a|, |m_a - m| > 2m_0 \text{ and } m_a < 0, \end{cases} \tag{32}$$

$$\frac{1+\epsilon}{\cos\theta_0} = (1 + f_x\cos\theta + v\sin\theta)/[2k\sin\phi(1 - k^2\sin^2\phi)^{1/2}], \tag{33}$$

$$\frac{d\theta}{d\bar{x}_0} = \frac{1+\epsilon}{\cos\theta_0}\left(\frac{d\theta_0}{d\bar{s}_0} + \bar{\kappa}\right), \tag{34}$$

$$\frac{d\bar{x}}{d\bar{x}_0} = \frac{1+\epsilon}{\cos\theta_0}\cos\theta, \tag{35}$$

$$\frac{d\bar{y}}{d\bar{x}_0} = \frac{1+\epsilon}{\cos\theta_0}\sin\theta, \tag{36}$$

$$\bar{x}(0) = \bar{y}(0) = \theta(p/l) = 0, \tag{37}$$

$$\bar{d} = \bar{y}(p/l) - 1 + 2\frac{E(k) - E(\rho, k)}{K(k) - F(\rho, k)}, \tag{38}$$

$$v_1 - \mu v_2 = 0, \tag{39}$$

$$\bar{d}_1 + \bar{R}_1(\sigma_1 f_x + \bar{\lambda}_1 v_1) + \beta[\bar{d}_2 + \bar{R}_2(\sigma_2 f_z + \bar{\lambda}_2 v_2)] = u. \tag{40}$$

After $\bar{x}_1$ and $\bar{x}_2$ are determined, the extensional strains in the warp and the weft directions as measured from the stress-free state are

$$\bar{\epsilon}_x = \frac{\bar{x}_1(p_1/l_1)}{p_1/l_1} - 1 \tag{41}$$

and

$$\bar{\epsilon}_z = \frac{\bar{x}_2(p_2/l_2)}{p_2/l_2} - 1 \tag{42}$$

respectively. Let us set the extensional strains in the initial state to be $\bar{\epsilon}_x = \bar{\epsilon}_{x_0}$ and $\bar{\epsilon}_z = \bar{\epsilon}_{z_0}$. The observed extensional strains are measured from the initial state. They are

$$\bar{\epsilon}_x = \frac{\bar{\epsilon}_x - \bar{\epsilon}_{x_0}}{1 + \bar{\epsilon}_{x_0}} \tag{43}$$

and

$$\epsilon_z = \frac{\bar{\epsilon}_z - \bar{\epsilon}_{z_0}}{1 + \bar{\epsilon}_{z_0}}. \tag{44}$$

NUMERICAL RESULTS

Our solutions of eqns (33)–(37) and (40) are based on a trial and error method. We first try a value for v and solve for θ, $\bar{x}$ and $\bar{y}$. It is found that if $\theta(0)$ is known, then θ, $\bar{x}$ and $\bar{y}$ can be solved numerically by the Runge–Kutta method. Here, we shall try a value for $\theta(0)$ and find $\theta(p/l)$. The correct value of $\theta(0)$ must make $\theta(p/l) = 0$. Again $\theta(0)$ can be found by a trial-and-error method. In our computation, we employed a modified Newton's method to find v and $\theta(0)$. We choose the values of f_x and f_z to increase from zero with constant increments. Hence the initial tried values of v and $\theta(0)$ can be estimated from their values in the previous steps by Lagrange's extrapolation formula.

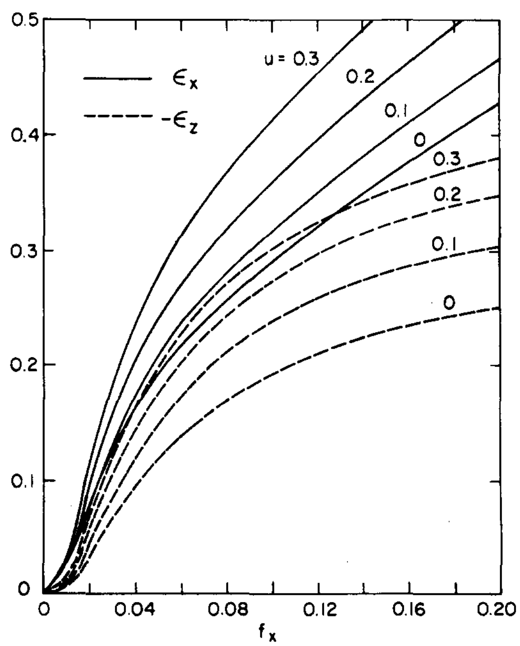

Fig. 3. ϵ_x and $-\epsilon_z$ curves for $\alpha_1 = 0.8$, $\alpha_2 = 0.7$, $n_1 = n_2 = 4$, $\sigma_1 = \sigma_2 = 0.4$, $\bar{R}_1 = \bar{R}_2 = 0.5$, $\bar{\lambda}_1 = \bar{\lambda}_2 = 0.3$, $\beta = 1$, $\mu = 1$, $\nu = 10$, $m_0 = 0.05$, $f_z = 0$ and various values of u.

Computations are carried out for a plain woven fabric with $\alpha_1 = 0.8$, $\alpha_2 = 0.7$, $n_1 = n_2 = 4$, $\sigma_1 = \sigma_2 = 0.4$, $\bar{R}_1 = \bar{R}_2 = 0.5$, $\bar{\lambda}_1 = \bar{\lambda}_2 = 0.3$, $\beta = 1$, $\mu = 1$, $\nu = 10$ and $m_0 = 0.05$. Two types of problems are considered here. In the first problem, the fabric is subject to a uniaxial extension in the x-direction. In this case, $f_z = 0$. In Fig. 3, ϵ_x is plotted vs f_x in solid lines for various values of u. When f_x is small, the elongation of the fabric is governed by both the extension and the bending of the yarn in the x-direction. When f_x is sufficiently large, the elongation is dominated by the extension of the yarn and the ϵ_x vs f_x curve would approach a straight line asymptotically. Note that $u = 0$ corresponds to the case of completely set fabric. It is seen from Fig. 3 that if the yarn geometry in the stress-free state remains identical, then the partially set fabric would in general have a larger extensional strain than the completely set fabric. Although it cannot be seen from Fig. 3, the partially set fabric has a slightly larger initial extensional modulus in comparison with the completely set fabric with the same geometry in the stress-free state. The strain in the transverse direction ϵ_z is negative. In Fig. 3, we plotted $-\epsilon_z$ vs f_x in dotted lines. When f_x approaches infinity, the yarns in the warp direction extends without any alteration of the yarn geometry in the weft direction. Hence all $-\epsilon_z$ curves approach asymptotically the horizontal lines.

In the second problem, the fabric is subjected to biaxial stresses of equal magnitude. Since $\alpha_1 \neq \alpha_2$, the extensional properties in the x and z directions are different. Consequently, the ϵ_x vs f_x curve and the ϵ_z vs f_z curve are also different. In this case, it is also found that the initial residual stresses can cause additional strain if the yarn geometries in the stress-free state remain identical.

CONCLUSIONS

(1) This study present a methodology for analyzing the problem of the finite biaxial extension of plain woven fabric. In order to make the problem realistic, the effect of initial stresses due to partial setting of yarns, loss in bending stiffness associated with fiber slippage in the yarn and the contact deformation of the yarns at the crimp are all included in the analysis.

(2) When the stress level is low, the extension of the fabric is governed by both the extension and the bending of yarns. When the stress level is high, the extension of the fabric is dominated by the extension of the yarn in the direction of the applied force.

(3) An additional tensile strain can be induced by the appearance of residual stresses in the yarn. The initial extensional modulus nevertheless remains practically constant.

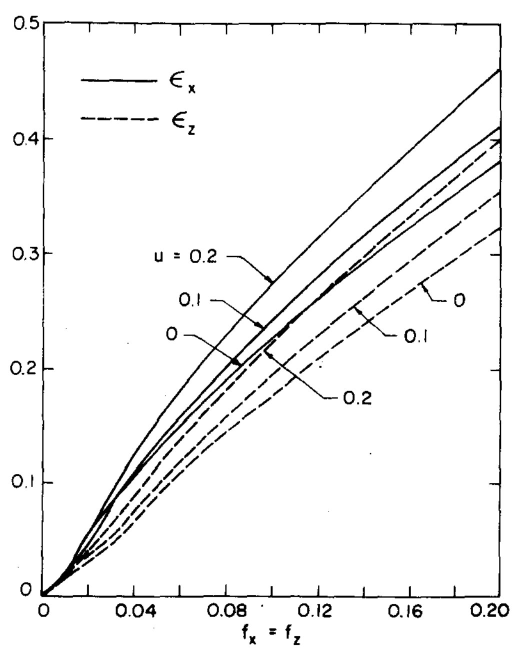

Fig. 4. ϵ_x and ϵ_z curves for $\alpha_1 = 0.8$, $\alpha_2 = 0.7$, $n_1 = n_2 = 4$, $\sigma_1 = \sigma_2 = 0.4$, $\bar{R}_1 = \bar{R}_2 = 0.5$, $\bar{\lambda}_1 = \bar{\lambda}_2 = 0.3$, $\beta = 1$, $\mu = 1$, $\nu = 10$, $m_0 = 0.05$, $f_x = f_z$ and various values of u.

(4) In the case of uniaxial extension of a fabric, contraction of the fabric in the transverse direction can be introduced through a variation in crimp height.

(5) In the case of biaxial extension of a fabric, different geometries of the yarns in the warp and the weft directions can cause a difference in the extensional properties in these directions.

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